# PHYX428-3 Spring 2009 : Quantum Field Theory 3

## Homework Assignment 4 : Pions as Goldstone Bosons

**Reading Assignment :** Peskin & Schroeder, Chapter 19.3 (first half)

**1.** QCD with Three Flavors of Quarks

Consider QCD with three quarks, $u$, $d$, $s$, invariant (aside from the quark masses) under $SU(3)_R \times SU(3)_L$. We define a $3 \times 3$ Goldstone boson field $U(x) = \text{Exp}\left[i\pi^a(x)T^a/f_\pi\right]$, with

$$\pi^a(x)T^a \;=\; \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \overline{K}^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{bmatrix}$$

where $\pi^0$ and $\eta^0$ are real fields and $\overline{K}^-$ is the complex conjugate of $K^+$ and $\overline{K}^0$ the conjugate of $K^0$.

**A.** Using the the spurion method we saw in class, derive the masses of all of the mesons (you can drop any terms of order $m_{u,d}^2/m_s$) .

**B.** Derive the Gell-Mann-Okuba relation,

$$3m_\eta^2 + 2m_{\pi^+}^2 - m_{\pi^0}^2 = 2m_{K^+}^2 + 2m_{K^0}^2 \ .$$

**C.** Devise an experimental determination of $m_u/m_s$ and $m_d/m_s$ using the properties of the pions.

**2.** A Ninth Goldstone boson?

In class we made the comment that there is no Goldstone corresponding to $U(1)_5$, the "axial baryon number" symmetry. Suppose for the moment that this symmetry was good and spontaneously broken by QCD, with a corresponding Goldstone boson $\xi(x)$.

**A.** Write $U(x) = \text{Exp}\left[i\pi^a(x)T^a/f_\pi + i\xi(x)/f_\xi\right]$ with $\pi^a T^a$ as above. Now the most general symmetry-preserving dimension 2 terms are,

$$\mathcal{L} = f_\pi^2 \ \text{Tr}\left[\partial_\mu U^\dagger \partial^\mu U\right] + F^2 \partial_\mu(\det\, U^\dagger)\partial^\mu(\det\, U)$$

Prove both terms are $SU(3)_R \times SU(3)_L \times U(1)_5$ invariant. Explain why didn't we write a term like the second one in class when we dealt with $SU(2)_R \times SU(2)_L$. Determine the constant $F$ in terms of $f_\pi$ and $f_\xi$ such that all of the Goldstones have canonically normalized kinetic terms.

**B.** Use the spurion method to determine the masses, but to simplify life slightly, set $m_u = m_d \ll m_s$. Identify the three lightest as the pions and show that there is another pseudo-Goldstone whose mass is $m \le \sqrt{3}m_\pi$. Historically, our failure to find such a particle is the "$U(1)$ problem" and was our first indication that something is funny with $U(1)_5$.